INTRINSICALLY KNOTTED GRAPHS WITH 21 EDGES

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ABSTRACT. We show that the 14 graphs obtained by ∇Y moves on K_7 constitute a complete list of the minor minimal intrinsically knotted graphs on 21 edges. We also present evidence in support of a conjecture that the 20 graph Heawood family, obtained by a combination of ∇Y and $Y\nabla$ moves on K_7, is the list of graphs of size 21 that are minor minimal with respect to the property not 2–apex.

1. INTRODUCTION

We say that a graph is **intrinsically knotted or IK** if every tame embedding of the graph in $\mathbb{R}^3$ contains a non-trivially knotted cycle. A graph is **minor minimal IK or MMIK** if it is IK, but no proper minor has this property. Robertson and Seymour's Graph Minor Theorem [RS] shows that there is a finite list of MMIK graphs. However, as it remains difficult to determine this list, research has focused on classification with respect to certain families of graphs. For example, it follows from Conway and Gordon's seminal paper [CG] that K_7 is the only MMIK graph on seven or fewer vertices; two groups [CMOPRW] and [BBFFHL] independently determined the MMIK graphs on eight vertices; and a classification of nine vertex graphs, based on a computer search, has been announced (see [Mo] and [GMN]). In terms of edges, it is known ([JKM] and, independently, [Ma]) that a graph of size 20 or less is not IK. The current paper presents a classification for graphs of 21 edges.

Kohara and Suzuki [KS] showed that the 14 graphs obtained from K_7 by a (possibly empty) sequence of ∇Y moves are MMIK. We will refer to this family as the **KS graphs**. Recall that a ∇Y **move** consists of deleting the edges of a 3-cycle abc of graph G, and adding a new degree three vertex adjacent to the vertices a, b, and c. The resulting graph G' has the same size as G and one additional vertex. Our main theorem asserts that the KS graphs are precisely the MMIK graphs of size 21.

Theorem 1.1. *The 14 KS graphs are the only MMIK graphs on 21 edges.*

As Kohara and Suzuki already proved these graphs are MMIK, our contribution is to show that no other graph of size 21 is IK. (Graphs of size 20 are not IK, so a connected 21 edge IK graph is also MMIK.)

We break the proof into cases by the order of the graph. Let G be a MMIK graph of size 21. We can assume $\delta(G)$, the **minimum degree**, is at least three. Indeed, deleting a degree zero vertex or contracting an edge of a vertex of degree one or two will result in an IK minor. Since a $(15, 21)$ graph must have at least

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one vertex of degree two or less, we can assume $|V(G)| \leq 14$. Our argument is an induction starting with the case of $(14, 21)$ graphs and descending to $(13, 21)$ and so on.

  Our induction on decreasing graph order relies on an observation essentially due to Sachs (see [S]): the ∇Y move preserves intrinsic knotting. The reverse $Y\nabla$ **move**, delete a degree three vertex and add edges to make its neighbors mutually adjacent, does not preserve IK and this is illustrated by the Heawood Family. Following [HNTY], **Heawood family** will denote the set of 20 graphs obtained from K_7 by a sequence of zero or more ∇Y or $Y\nabla$ moves. The family is illustrated schematically in Figure 1 (taken from [GMN]) where K_7 is graph 1 at the top of the figure and the $(14, 21)$ Heawood graph is graph 18 at the bottom. In addition to the 14 KS graphs, the Heawood family includes six additional graphs (graphs 9, 14, 16, 17, 19, 20 in the figure) that are not IK, as was shown independently in [GMN] and [HNTY]. Thus, for example, the $Y\nabla$ move from graph 5 to 9 takes an IK graph to one that is not. (That $Y\nabla$ does not preserve IK was first observed by Flapan and Naimi [FN]).
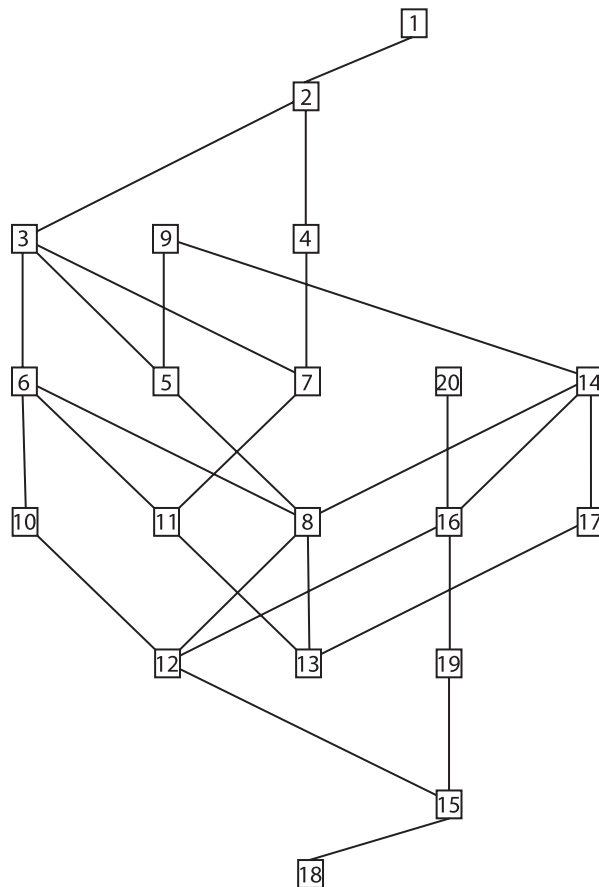

FIGURE 1. The Heawood family (figure taken from [GMN]). Edges represent ∇Y moves.

We conclude this introduction with some observations and questions about 21 edge graphs that are not 2–apex. Recall that a graph is n–**apex** if it can be made planar through deletion of n or fewer vertices (and their edges). Thus, a graph G is **not 2–apex, or N2A** if whenever two vertices (and their edges) are deleted, the resulting graph is not planar. We will make much use of the following lemma, which is a consequence of the observation, due independently to [BBFFHL] and [OT], that the join, $H * K_2$, of H and K_2 is IK if and only if H is nonplanar.

Lemma 1.2. [BBFFHL, OT] *If G is IK, then G is N2A.*

In other words the class of IK graphs is a subset of those that are N2A. In particular, every graph in the Heawood family is N2A and it's natural to ask if there are other 21 edge examples. (Since size 20 graphs are 2–apex [Ma], a connected 21 edge N2A graph is necessarily minor minimal or MMN2A.)

Question 1.3. *Is the Heawood family the set of graphs of size 21 that are MMN2A?*

The following observation allows us to answer the question for graphs of order ten or less.

Proposition 1.4. *Let G be a graph with either $|V(G)| \leq 8$ or else $|V(G)| \leq 10$ and $|E(G)| \leq 21$. If G is N2A and a $Y\nabla$ move takes G to G', then G' is also N2A.*

On the other hand, it is straightforward to settle the question for order 14 or more using the idea that a MMN2A graph has minimal degree 3. Thus, all that remains are graphs of orders 11, 12, or 13.

Proposition 1.5. *If G is MMN2A with $|E(G)| = 21$ and $|V(G)| \neq 11, 12, 13$, then G is a Heawood graph.*

In [HNTY], the authors show that the Heawood graphs are minor minimal with respect to the property intrinsically knotted or completely 3–linked. This suggests that property may be related to N2A.

Question 1.6. *How are N2A graphs related to those that are intrinsically knotted or completely 3–linked?*

It's easy to see that $Y\nabla$ does not preserve N2A in general. For example, the disjoint union of three $K_{3,3}$ graphs is N2A, but applying a $Y\nabla$ move destroys this property. However, it may be that Proposition 1.4 can be extended to all graphs of size 21.

Question 1.7. *Does $Y\nabla$ preserve N2A on graphs of size 21?*

As $Y\nabla$ does preserve N2A in the Heawood graphs, an affirmative answer to Question 1.3 would imply the same for Question 1.7. If so, we could ask about the first instance of $Y\nabla$ not preserving N2A.

Question 1.8. *What is the simplest (e.g., smallest in size or order) graph G that is N2A but admits a $Y\nabla$ move to a graph G' that is 2–apex?*

After introducing some preliminary lemmas in the next section, we devote one section each to intrinsic knotting of graphs of order 14, 13, 12, 11, and 10, respectively. As graphs of order nine or less were treated earlier in [Ma], taken together this constitutes a proof of Theorem 1.1. We conclude the paper with a proof of Propositions 1.4 and 1.5 in Section 8.

After preparing this paper we learned that Lee, Kim, Lee, and Oh [LKLO] have also announced a proof of Theorem 1.1. Our approach is based on the first author's thesis [B].

2. Definitions and Lemmas

As mentioned in the introduction, we prove the main theorem by induction starting with graphs of 14 vertices and working down to those having ten. We begin by observing that it is enough to consider triangle–free graphs.

Remark 2.1. *As the KS graphs are precisely the IK graphs in the Heawood family, it will be enough for us to show that size 21 MMIK graphs are Heawood. Using our induction, this allows us to reduce to the case of triangle–free graphs. Indeed, if a 21 edge MMIK graph G has a triangle, apply a ∇Y move to obtain an IK graph G' with one additional vertex. This graph must be MMIK as graphs on 20 edges are not IK. Then, by the inductive hypothesis, G' is Heawood, whence G is also.*
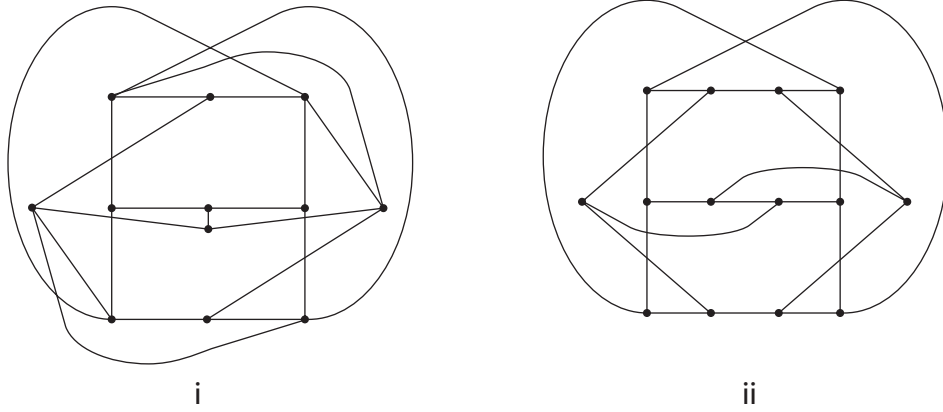

FIGURE 2. The two triangle–free Heawood graphs. i) H_{12} ii) C_{14}

Note that only two graphs in the Heawood family are triangle–free, namely graphs 13 and 18 in Figure 1. These graphs were named H_{12} and C_{14} by Kohara and Suzuki [KS], see Figure 2.

Throughout this paper, for $a, b \in V(G)$, we will use $G - a$ and $G - a, b$ to denote the induced subgraphs on $V(G) \setminus \{a\}$ and $V(G) \setminus \{a, b\}$, respectively. We will also write $G + a$ to denote a graph with vertices $V(G) \cup \{a\}$ that includes G as the induced subgraph on $V(G)$. In case $V(G)$ and $\{a\}$ are included in the vertex set of some larger graph, $G + a$ will mean the induced subgraph on $V(G) \cup \{a\}$.

Here is an example of how Lemma 1.2 and the triangle–free condition work in concert. We'll use $|G|$ to denote the order, or number of vertices of a graph.

Lemma 2.2. *Let G be a graph with minimum degree $\delta(G) \geq 3$. Suppose $\exists a, b \in V(G)$ such that $G - a, b$ has a tree component T. If $|T| \leq 3$ or T has a degree two vertex adjacent to a leaf, then G has a triangle.*

Proof. Since $\delta(G) \geq 3$, leaf vertices of T are adjacent to both a and b in G while degree two vertices are adjacent to at least one. Thus, under the hypotheses on T,

a triangle is formed, with a leaf of the tree and one of a and b constituting two of the triangle's vertices. $\square$

In other words, if G is MMIK, triangle-free, and of order 21, then, by Lemma 1.2, G is N2A. So $\forall a, b \in V(G)$, $G - a, b$ is nonplanar. Lemma 2.2 restricts the structure of any tree components of these nonplanar graphs. Our strategy is to combine enough restrictions of this type to either force a contradiction or else demonstrate that G is H_{12} or C_{14}.

As above, we will often encounter non-planar graphs of the form $G - a, b$. Although this means $G - a, b$ has either a K_5 or $K_{3,3}$ minor by Kuratowski's theorem, the $K_{3,3}$ case is more important in our argument. Especially, we will often encounter **split $K_{3,3}$'s**, graphs obtained from $K_{3,3}$ by a finite (possibly empty) sequence of vertex splits. Conversely, this means that starting from a split $K_{3,3}$ graph G, we can recover a $K_{3,3}$ minor by repeatedly deleting vertices using the following two **deletion operations** until there remain no vertices of degree less than three.

   D1: Delete a vertex of degree one and its edge.
   D2: Delete a degree two vertex b replacing its edges ab and bc with a new
          edge ac.

We will refer to the six vertices in G that survive this sequence of deletions as the **orginal vertices.** Since there may be more than one sequence of deletion moves leading to $K_{3,3}$, in general, there's more than one way to choose original vertices. As our argument does not depend on this choice, we'll often assume, without further explanation, that a specific choice has been made. An **original 4–cycle** is a cycle C in G that passes through exactly four original vertices. The **split 4–cycle** of C is the component of C in $G - v, w$ where v and w are the two original vertices not in C.

In addition to D1 and D2 we will have occasion to refer to D0, meaning deletion of an isolated vertex. The **simplification** of a graph G is the graph G' with $\delta(G') \geq 3$ formed by repeatedly applying the three deletion operations to G. Although, in principle, G' may be a multi-graph (e.g., applying D2 to a vertex in a three cycle will lead to a double edge), that won't happen in the examples we consider in this paper. For example, as mentioned above, the simplification of a split $K_{3,3}$ is $K_{3,3}$.

FIGURE 3. The simplification of a split $K_{3,3}$ relative to a.

We'll also use the idea of simplification relative to a vertex. Let G be a split $K_{3,3}$ and $a \in V(G)$. The **simplification of G relative to a**, $G|_a$, is the graph formed

by repeatedly applying D1 and D2 to delete vertices other than a until all vertices (except possibly a) have degree at least three. Then either a is an original vertex or else $G|_a$ is one of the three graphs of Figure 3. In case $a = v_1$ is an original vertex of G or we have the graph of Figure 3i, we say that v_1 is the **nearest part** of $K_{3,3}$ to a. In the case of Figure 3ii or iii, we will say that the edge v_1w_1 is the **nearest part** of $K_{3,3}$ to a.

Here's an alternative characterization of split $K_{3,3}$ graphs. We use $\chi(G) = |G| - \|G\|$ to denote the **Euler characteristic of a graph**, that is, the difference between the $|G| = |V(G)|$ and $\|G\| = |E(G)|$.

Lemma 2.3. *A graph G is a split $K_{3,3}$, if and only if, it is connected with a $K_{3,3}$ minor and $\chi(G) = -3$.*

Proof. Assume G is a split $K_{3,3}$. Since G can be made using a series of vertex splits on a $K_{3,3}$ graph, then it is connected and has a $K_{3,3}$ minor. Since each vertex split adds exactly one vertex and one edge, $\chi(G) = \chi(K_{3,3}) = -3$.

Now assume G has a $K_{3,3}$ minor, is connected, and that $\chi(G) = -3$. So G can be built by adding vertices and edges to a split $K_{3,3}$, H. Since $\chi(G) = -3 = \chi(H)$ an equal number of vertices and edges are added. And as G and H are both connected, we can build G from H through a sequence of connected graphs as follows. At each step we add a vertex along with one of its edges so as to connect the new vertex to the connected graph of the previous step. In other words, G is obtained from H by a series of vertex splits. Thus, G is also a split $K_{3,3}$. $\square$

Starting from G MMIK, Lemma 1.2 implies that every $G - a, b$ is non-planar. The next two lemmas show that, if $G - a, b$ is a split $K_{3,3}$ then a and b both must have independent paths to each of the original vertices.

Lemma 2.4. *Let G be a split $K_{3,3}$. The graph $G + a$ is 1–apex if there is an original vertex, v, such that every path from a to v contains another original vertex.*

Proof. Consider $G + a$ where G is a split $K_{3,3}$, and say that $v \in V(G)$ is an original vertex such that any path from a to v contains another original vertex. Let w be an original vertex that is adjacent to v in the underlying $K_{3,3}$. If a is adjacent to $b \in V(G - v, w)$, then either b is on the split 4-cycle in $G - v, w$ or else b is a vertex that has w as its nearest part in the underlying $K_{3,3}$. It follows that $(G + a) - w$ is planar and $G + a$ is 1-apex. $\square$

Lemma 2.5. *Suppose G is N2A and $G^* = G - a, b$ is a split $K_{3,3}$ for some $a, b \in V(G)$. Then, in $G^* + a$, for each original vertex v, there is a path from a to v that avoids the other original vertices, and similarly for $G^* + b$.*

Proof. Since G is N2A, $G^* + a$ is not 1-apex. Apply Lemma 2.4. $\square$

We conclude the introduction by characterizing graphs formed by adding a vertex of degree three or four to a split $K_{3,3}$.

Lemma 2.6. *If $G + a$ is formed by adding a degree three vertex a to a split $K_{3,3}$ graph G and $G + a$ is not 1-apex, then the simplification of $G + a$ is the graph of Figure 4.*

Proof. By Lemma 2.4, there are paths from a to each original vertex that avoid all other original vertices. Let $N(a) = \{n_1, n_2, n_3\}$. As there are six vertices and $d(a) = 3$, then each n_i must have an edge as its nearest part, and up to relabeling
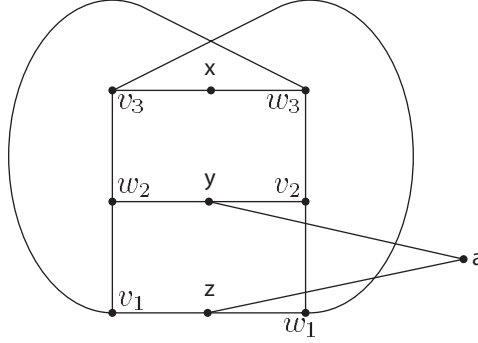
FIGURE 4. Adding a degree 3 vertex to a split $K_{3,3}$.

of the original vertices, n_i has the edge $v_i w_i$ of G as its nearest part. This means the simplification of $G + a$ is the graph of Figure 4. □

Lemma 2.7. *If $G + a$ is formed by adding a vertex a of degree four to a split $K_{3,3}$ graph G and $G + a$ is not 1-apex, then $G + a$ is one of the seven graphs in Figure 5.*

Proof. By Lemma 2.4, there are paths from a to each original vertex that avoid all other original vertices. Let $N(a) = \{n_1, n_2, n_3, n_4\}$. As there are six vertices and $d(a) = 4$, then there is an n_i, say n_1, that has an edge, say $v_1 w_1$, as its nearest part. Since there are four original vertices left and three neighbors of a, another n_i, say n_2, must have an edge as its nearest part with vertices disjoint from $\{v_1, w_1\}$, call it $v_2 w_2$. There are three graphs generated when a has a neighbor whose nearest part is an original vertex of G and four more when a has no such neighbor. Figure 5 shows the graphs that results from this condition. □

3. 14 VERTEX GRAPHS

In this section, we will show that the only 14 vertex MMIK graph on 21 vertices is the KS graph C_{14} (Figure 2ii). See [KS] for the names, such as C_{14}, of the KS graphs. We first characterize N2A graphs.

Proposition 3.1. *Let G be a connected $(14, 21)$ graph. If G is N2A, then G is the KS graph C_{14}.*

Proof. Let G be a connected $(14, 21)$ graph and assume G is N2A. If a 21 edge graph G has $\delta(G) < 3$ then, by applying a deletion operation, G simplifies to a graph with fewer than 21 edges and is therefore 2-apex. Since $14 \times 3 = 2 \times 21$, G must have the degree sequence (3^{14}). For any vertex a, $G - a$ has degree sequence $(3^{10}, 2^3)$. Now choose another vertex, b, such that $G^* = G - a, b$ has the sequence $(3^6, 2^6)$ (i.e., a and b have no common neighbors). There are enough degree 3 vertices in $G - a$ to assure we can always choose such a b.

Since G is N2A and G^* has the sequence $(3^6, 2^6)$, then G^* must be a split $K_{3,3}$. By Lemma 2.6, $G^* + a$ simplifies to Figure 4. Then $G' = (G^* + a) - w_3$ is another split $K_{3,3}$.

By Lemma 2.6, b must have a path to a that avoids v_3, w_1, w_2, y and z. Since a and b have no common neighbors, this means b has a neighbor b_1 that is adjacent
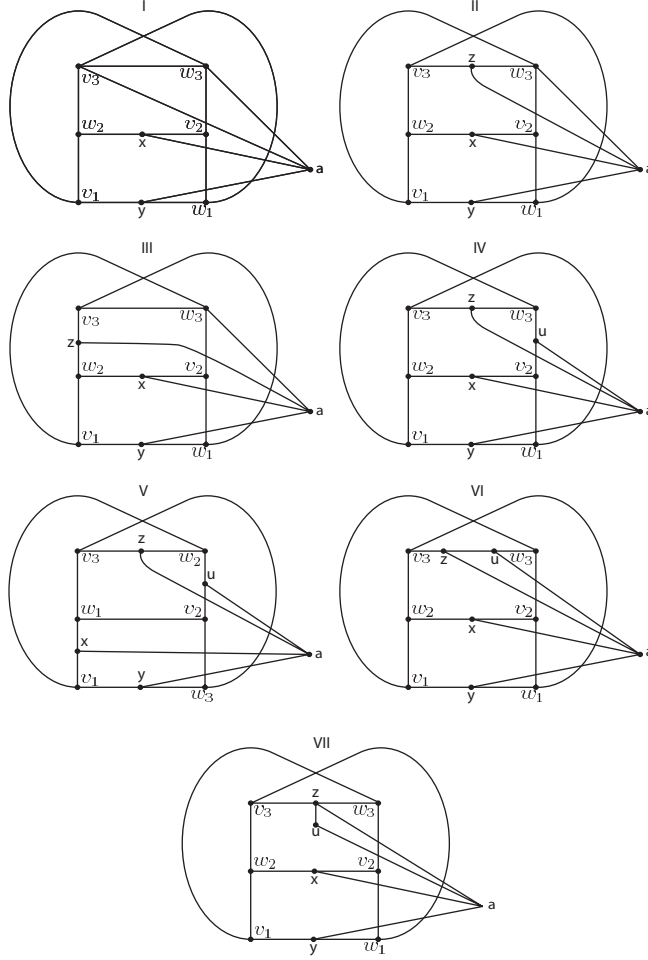
FIGURE 5. Adding a degree 4 vertex to a split $K_{3,3}$.

to x. So, there are two cases: in $G' + b$, either b_1 is of degree two, or else it has v_3 as a third neighbor. (See Figure 6.)

In either case, b_1 gives paths from b to the original vertices a and v_3 and there are three ways to split the remaining four original vertices into two pairs. However, we see that $G - w_2, z$ is planar (and G is 2–apex), unless we make the choices shown in Figure 6. In both cases, adding w_3 back will give us C_{14}. Hence the only connected (14,21) graph that is N2A is C_{14}. $\qquad\square$

Corollary 3.2. *The only MMIK* $(14, 21)$ *graph is the KS graph* C_{14}.

4. 13 VERTEX GRAPHS

Proposition 4.1. *The only MMIK* $(13, 21)$ *graph is the KS graph* C_{13}.
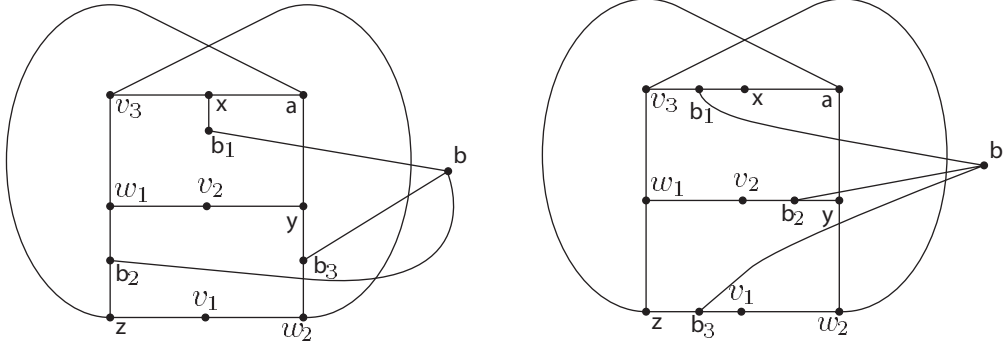
FIGURE 6. Two possibilities for $G' + b$.

Proof. Let G be an MMIK $(13, 21)$ graph. As in Remark 2.1, if G has a triangle, it must be C_{13}, which is MMIK. So, we will assume G is triangle-free and force a contradiction (generally, by arguing G must have a triangle).

An MMIK graph G will have $\delta(G) \geq 3$ and one of the following three degree sequences: $(3^{12}, 6)$, $(3^{11}, 4, 5)$, or $(3^{10}, 4^3)$.

Case 1: $(3^{12}, 6)$

Assume G has degree sequence $(3^{12}, 6)$. Delete a and b not adjacent with $d(a) = 6$, $d(b) = 3$. Then $\|G - a, b\| = 12$ and by [Ma] if $G - a, b$ is not planar, it has a K_2 component, which, as in Lemma 2.2 results in a triangle in G, a contradiction.

Case 2: $(3^{11}, 4, 5)$

Assume G has degree sequence $(3^{11}, 4, 5)$. Delete the degree five and four vertices a and b. If a and b are not adjacent, then, as in the previous case, G has a triangle. So, we can assume a and b are adjacent. Then $\|G - a, b\| = 13$ and by [Ma] if $G - a, b$ is not planar, it is either $K_5 \cup K_2 \cup K_2 \cup K_2$, in which case G has a triangle, or has a component with $K_{3,3}$ minor as well as at least one tree component. However, a leaf of a tree component will form a triangle with a and b. In either case, we deduce that G has a triangle, a contradiction.

Case 3: $(3^{10}, 4^3)$

Assume G has degree sequence $(3^{10}, 4^3)$. Delete two degree four vertices a and b. Assume a and b are not adjacent; then, $\|G - a, b\| = 13$ and by [Ma] if $G - a, b$ is not planar, it is either $K_5 \cup K_2 \cup K_2 \cup K_2$, in which case G has a triangle, or has a component with $K_{3,3}$ minor as well as at least one tree component, T. If $|T| \leq 3$ then, by Lemma 2.2, G has a triangle, which is a contradiction. So we'll assume $|T| > 3$ and we have two cases: $|T| = 4$ or $|T| = 5$. (There are at least six vertices in the $K_{3,3}$ component, so at most five of the 11 vertices in $G - a, b$ left over for T.)

Since $\chi(G - a, b) = -2$, there are exactly two components, T and the component with $K_{3,3}$ minor, call it H. Moreover, H is connected with $\chi(H) = -3$ and, therefore, a split $K_{3,3}$ by Lemma 2.3. If T is not a star, Lemma 2.2 shows that there is a triangle, which is a contradiction. So we'll assume T is a star.

If $|T| = 4$ then a must be adjacent to all three of its leaves. Let v be an original vertex of H. The fourth neighbor of a is either the fourth vertex of T or a vertex in H. In either case, $G - b, v$ is planar and G is 2–apex, hence not IK.

If $|T| = 5$ both a and b are adjacent to the four leaf vertices of T and have no neighbors in H. So G is not connected and thus not MMIK, again, a contradiction.

Say that a graph G has this sequence but there does not exist a pair of degree four vertices, a and b, such that a and b are not adjacent. Then the three degree four vertices form a triangle in G, which is a contradiction. This completes the argument for Case 3 and with it the proof of the proposition. $\square$

5. 12 VERTEX GRAPHS

Proposition 5.1. *The only MMIK* $(12, 21)$ *graphs are the KS graphs* C_{12} *and* H_{12}.

Proof. Suppose G is a MMIK $(12, 21)$ graph. As in Remark 2.1, if G has a triangle, it must be C_{12}. So we'll assume that G is triangle-free and either show G is H_{12} or else deduce a contradiction. Note that H_{12} has degree sequence $(3^6, 4^6)$

Let us first consider a $(12, 21)$ MMIK graph G such that there exists a pair of vertices a and b with $\|G - a, b\| < 13$. By an Euler characteristic argument, if $G - a, b$ is nonplanar, then it contains at least one tree, T, and $|T| \leq 4$. By Lemma 2.2, unless T is a star on four vertices, this means G has a triangle, which is a contradiction. So we will assume that in the graph $G - a, b$, the tree component T has order four and is a star. This implies that the other component must be the graph $K_{3,3}$. Adding the vertex a back into the graph, we see that a needs to be adjacent to all the leaves of T. Also, by Lemma 2.5, a must be adjacent to every vertex in the $K_{3,3}$. Hence $d(a) \geq 9$ and $\|G - b\| \geq 21$ which is impossible. So, if $G - a, b$ has size less than 13, we have a contradiction.

This helps us narrow down the degree sequences we have to consider. For instance, suppose G is a $(12, 21)$ graph and has a vertex a, such that, $d(a) > 5$. Then there is another vertex b, such that, b is not adjacent to a and $d(b) > 2$, so $\|G - a, b\| < 13$, leading to a contradiction, as above. Similarly, if there is a vertex of a of degree five and another vertex b such that either $d(b) = 5$, or else $d(b) = 4$ and b is not adjacent to a, then again $\|G - a, b\| < 13$ and we have a contradiction. Recall that G MMIK implies $\delta(G) \geq 3$. In order to avoid a triangle among vertices of degree four or more, it remains only to consider the two cases where G has the degree sequence $(3^7, 4^4, 5)$ or $(3^6, 4^6)$.

Case 1: $(3^7, 4^4, 5)$

Assume G has degree sequence $(3^7, 4^4, 5)$. Denote the vertex of degree five as a and recall that it must be adjacent to all the vertices of degree four. Delete a and note that $G - a$ has the sequence $(2, 3^{10})$. Next, delete b such that the degree of b in G was four. Notice that if b is adjacent to the degree two vertex in $G - a$, that would imply a triangle in G, so we assume it is not. Then $G - a, b$ has the degree sequence $(2^4, 3^6)$. If $G - a, b$ is nonplanar then, since $\chi(G - a, b) = -3$, it contains a split $K_{3,3}$ and if it is not connected, its other component is a cycle of order 3 or 4.

As a has degree five in G and b is one of its neighbors, a has four neighbors in $G - a, b$, exactly one of them being a vertex of degree two. This means, in contradiction to Lemma 2.5, there is at least one original vertex of the split $K_{3,3}$ that has no path to a that avoids the other original vertices. The contradiction shows there is no such graph with degree sequence $(3^7, 4^4, 5)$

Case 2: $(3^6, 4^6)$

This case is hard as H_{12} has this degree sequence and H_{12} is a triangle-free MMIK graph. We can continue to eliminate many cases that result in a triangle,

but in the end we will need to explicitly show that a MMIK graph with this sequence is H_{12}. We will delete vertices a and b both of degree four, which we can assume to be nonadjacent. Let $G^* = G - a, b$. Notice that $\chi(G^*) = -3$ and we have two cases: either G^* is connected or it is not.

Assume that G^* is nonplanar and is not connected. Using Lemma 2.2, G^* is either a nonplanar $(7, 10)$ graph together with a cycle of order three, a nonplanar $(6, 9)$ graph together with a cycle of order four, or a nonplanar $(6, 10)$ graph together with a star of order four. In the first case, a cycle of order 3 is a triangle in G, a contradiction. In the case where G^* has a cycle of order 4, denote it by C, then the other component is $K_{3,3}$. Since one of a and b has at least two neighbors on C, adding that vertex and deleting any vertex of the $K_{3,3}$ in G^* results in a planar graph meaning G is 2–apex, contradicting G MMIK. Finally, if G^* has a star of order 4, then both a and b are adjacent to each of the three leaves of the star. The nonplanar $(6, 10)$ has minimal degree one or more and must be a $K_{3,3}$ with an extra edge, call it $v_1 v_2$. Then $G - a, v_1$ is planar and G is 2–apex, contradiction. Thus, we conclude that if G^* is not connected, then G is not MMIK.

We will now assume that G^* is connected. So by Lemma 2.3, G^* is a split $K_{3,3}$. Then by Lemma 2.7, we see that $G^* + a$ and $G^* + b$ simplify to one of the seven graphs in Figure 5. We shall denote our graphs as in the figure and use the vertex labels given there for convenience.

Notice that in the cases of VI and VII, $G^* + a$ is the graph shown (no additional vertex splits are needed) and G has a triangle, a contradiction. So we will assume that $G^* + a$ (and, similarly, $G^* + b$) simplify to one of the other five graphs. If $G^* + a$ simplifies to V, then $|G^* + a| = 11$, so we do not have any additional vertex splits (and $G^* + a$ is as shown). Deleting the vertices labeled z and x in the figure, the resulting graph has a planar representation. Furthermore, if b is not a neighbor of a, b can be a neighbor to all the other remaining vertices and maintain the graph's planarity. Since our assumption was that a and b are not neighbors, we have shown that G is 2-apex in the case where $G^* + a$, or, by symmetry, $G^* + b$, simplifies to graph V in Figure 5.

Going on to the next possibility, assume $G^* + a$ simplifies to IV. Since $|G^* + a| = 11$ we do not have any vertex splits. If b is not a neighbor of y, then $G - v_1, w_1$ is planar. So assume that y and b are adjacent in G. If b is not adjacent to x or if b is not adjacent to z then $G - y, z$ and $G - y, x$ are planar respectively. Thus b will have x, y, and z as neighbors. If its fourth neighbor is not u, then G will have a triangle. This shows that both a and b will have x, y, z, and u as neighbors. But then $G - y, x$ is planar. So, if $G^* + a$ or $G^* + b$ simplifies to IV in Figure 5, then G is not MMIK

Considering the case where $G^* + a$ simplifies to to III in Figure 5, we notice that III has ten vertices and $G^* + a$ has eleven vertices. This implies that $G^* + a$ is III with a vertex split. We will denote the vertex created by this split u and refer to u as the **vertex split**. (In other words, much as the deletion moves D1 and D2 allow us to imagine edge contractions as vertex deletions, we tend to think of a vertex split in terms of adding a vertex.) Notice that deleting w_3 and z from $G^* + a$ gives us a planar graph, unless both a and b have u as a neighbor. Assume u is a neighbor of both a and b and recall that $G^* + b$ simplifies to one of graph I, II, or III. We can rule out II, since that would require another vertex split. We then see that in the graph $G^* + b$, the neighborhood of b, after deleteing u (by deletion

move D2), is $\{x, y, z, w_3\}$, $\{x, y, v_3, w_3\}$, or $\{x, z, v_2, w_3\}$. In all of these cases, if we choose to delete x and w_3 we will get a planar graph even if we add a and b back in, since they both have u as a neighbor. Hence, in the case where $G^* + a$ or $G^* + b$ simplifies to III in Figure 5, G is 2-apex.

Next suppose $G^* + a$ simplfies to graph II in Figure 5. Notice, as when we considered graph III, there is a vertex split, u, on G^*a not shown in II. In II, we see that the vertices z, a, and w_3 form a triangle, so we need only consider the graphs for which u is on one of the edges of this triangle. Assume u is between z and b. We see that u is a neighbor of both a and b. If $G^* + b$ is topologically equivalent to graph II, either G contains a triangle (a contradiction), or else the neighborhood of b is $\{u, x, y, w_3\}$ or $\{u, x, y, v_3\}$. For both choices of b's neighborhood, $G - w_3, v_3$ is planar. So we assume that $G^* + b$ simplifies to graph I in Figure 5. Then, b is adjacent to u and u is adjacent to z, so b is adjacent to x or y. Without losing generality, we can say that b is adjacent to x. Hence, b also has w_1 and v_1 as neighbors. Clearly, $G - w_1, v_1$ is planar.

We shall now assume that u is between z and w_3. Again, u is adjacent to b. Not considering cases that would give us triangles, b has the neighborhood $\{u, y, x, v_3\}$ if $G^* + b$ simplifies to II in Figure 5, or else b has the neighborhood $\{u, y, w_2, v_2\}$ or $\{u, x, w_1, v_1\}$ if $G^* + b$ simplifies to I. The graphs $G - w_3, v_3$, $G - w_2, v_2$, and $G - w_1, v_1$ are planar in each of these respective cases.

Next, suppose u is between w_3 and a. Notice that b is adjacent to u, so whether $G^* + b$ simplifies to graph I or II in Figure 5, b will have x and y as neighbors. Thus $G - w_3, v_3$ is planar. Thus, when $G^* + a$ or $G^* + b$ simplifies to II in Figure 5, we arrive at a contradiction.

Lastly, we approach the case where $G^* + a$ simplifies to graph I in Figure 5. Notice again the triangle formed between a, w_3, and v_3, implies there is a vertex split, denote it by z, on an edge of the triangle. Obviously, the cases where z is between a and v_3 and between a and w_3 are symmetric to one another. We next show that they are also symmetric to placing z in between v_3 and w_3. Indeed, let H_I denote the graph of Figure 5I. Note that $H_I - a$ and $H_I - v_3$ are isomorphic and the identification extends to an isomorphism of H_I that interchanges a and v_3. This isomorphism shows that a z on $v_3 w_3$ is symmetric to one on $a w_3$. So, without loss of generality, we will assume that z is between v_3 and w_3.

We still have another vertex split, u, somewhere on our graph. If we delete v_3 and w_3, we notice that as long as both a and b are not both adjacent to u, then the graph is planar. Vertex b is adjacent to z because z has degree 3 in G and b is also adjacent to u, which is is a neighbor of x or y since $G^* + b$ is topologically equivalent to I. In either case b is also adjacent to v_1 and w_1 or v_2 and w_2 respectively and the graph formed is H_{12} (see Figure 2).

Therefore, if G is a MMIK $(12, 21)$ and has no triangle, it will be H_{12}.   □

6. 11 VERTEX GRAPHS

Proposition 6.1. *The only MMIK $(11, 21)$ graphs are the KS graphs, H_{11}, E_{11}, and C_{11}.*

Proof. As in the previous sections, we will use that if an $(11, 21)$ graph G is MMIK, then it has a minimum degree of at least three, and, following Remark 2.1, we assume G is triangle–free and look for a contradiction.

By Lemma 1.2, G is 2–apex and no $G - a, b$ is planar. So assume that we delete two vertices, a and b, and in the process we also delete at least ten edges. The resulting graph $G - a, b$ has order $|G - a, b| = 9$ and size $\|G - a, b\| \leq 11$ and a minimum degree of at least one. Thus $\chi(G - a, b) \geq -2$. Since $\chi(K_5) = -5$ then our graph cannot have a K_5 minor since that would require at least three trees and we do not have enough vertices. (Since $\delta(G - a, b) \geq 1$, a tree has at least two vertices.) If $G - a, b$ is non-planar it must have a $K_{3,3}$ minor. Now, $\chi(K_{3,3}) = -3$ so $G - a, b$ will have at least one tree, which must be of order two or three. By Lemma 2.2 this means G has a triangle, which is a contradiction. So there can be no pair of vertices a and b that result in the deletion of ten or more edges.

There are six degree sequences that satisfy this condition on the deletion of two vertices: $(6, 4^6, 3^4)$, $(5^4, 4, 3^6)$, $(5^3, 4^3, 3^5)$, $(5^2, 4^5, 3^4)$, $(5, 4^7, 3^3)$, and $(4^9, 3^2)$.

Case 1: $(6, 4^6, 3^4)$

Assume the graph G has $(6, 4^6, 3^4)$ as its degree sequence. Notice that if $\nexists a, b$ such that $\|G - a, b\| \leq 11$, then the vertex of degree six is a neighbor of each vertex of degree four. Since there must be a pair of adjacent degree four vertices in G, then G has a triangle, a contradiction.

Case 2: $(5^4, 4, 3^6)$ and $(5^3, 4^3, 3^5)$

Assume the graph G has either the degree sequence $(5^4, 4, 3^6)$ or $(5^3, 4^3, 3^5)$. It's apparent that if we cannot delete an a and b from G such that $G - a, b$ has 11 edges, then all the vertices of degree five are mutually adjacent. Hence there is a triangle in G, a contradiction.

Case 3: $(5^2, 4^5, 3^4)$ and $(5, 4^7, 3^3)$

Assume that G has either $(5^2, 4^5, 3^4)$ or $(5, 4^7, 3^3)$ as its degree sequence. We choose to delete two vertices a and b such that the degree of b is 5, the degree of a is 4, and b is not a neighbor of a. It may not be immediately obvious why we can choose such an a and b for the degree sequence $(5^2, 4^5, 3^4)$; however, if b is a neighbor to all the vertices of degree 4, then the two vertices of degree 5 are not neighbors, so we can have a $(9, 11)$ graph with the deletion of the two degree five vertices. As discussed above, this leads to a triangle in G.

So, we can delete vertices a and b that are not adjacent and of degree four and five. This means that $G - a, b$ is a $(9, 12)$ graph, so $\chi(G - a, b) = -3$. If $G - a, b$ is nonplanar and disconnected, then it has either a K_5 minor or a $K_{3,3}$ minor with an additional component of order at most three. Whether this component is a tree or a cycle does not matter since either way it will imply a triangle in G. So, we'll assume that $G - a, b$ is connected.

Denote $G - a, b$ as G^*. Since G^* is connected and $\chi(G^*) = -3$, if it is nonplanar then it has a $K_{3,3}$ minor, and hence, by Lemma 2.3, G^* is a split $K_{3,3}$. Using Lemma 2.7 and the restriction that G has only 11 vertices, we see that $G^* + a$ simplifies to one of the graphs I, II, or III in Figure 5. Notice that II automatically implies a triangle in G. If G^* simplifies to III, then deleting v_1 and w_1, v_2 and w_2, or v_3 and w_3 respectively, shows us that b has y, x, and z as neighbors as G is not 2-apex. Since b is of degree five and does not have a as a neighbor, then adding it back in will create a triangle in G.

If $G^* + a$ simplifies to I we notice that there must another vertex split, z, on one of the edges on the triangle formed by a, v_3, and w_3. If z is between v_3 and w_3 then $G - v_3, w_3$ is planar. Having z between a and v_3 or a and w_3 are symmetric cases, so we will assume z is between a and w_3. Since b is adjacent to z, if b has w_3

as a neighbor there is a triangle. If not, since any four of the other seven possible neighbors of b will include at least two neighboring vertices, G will have a triangle.

We conclude that if G cannot have $(5^2, 4^5, 3^4)$ or $(5, 4^7, 3^3)$ as its degree sequence.

Case 4: $(4^9, 3^2)$

This degree sequence can be considered the hard case for $(11, 21)$ graphs since the maximum number of edges we can take away with the deletion of two vertices is 8. In that case, $G - a, b$ has 9 vertices and 13 edges and there are many such nonplanar graphs. So we will apply a slightly different method for this case. Assuming that G has the degree sequence $(4^9, 3^2)$ we first notice that together, the vertices of degree three have at most six neighbors. Hence, there is a vertex of degree four, denote it by v, whose neighbors are all vertices of degree four. If any of the neighbors of v are mutually adjacent, then G has a triangle. Deleting all four neighbors of v gives us a $(7, 5)$ graph, G^*, that has at least one vertex of degree zero. Also, since G has maximum degree four then G^* also has maximum degree four. Since $\chi(G^*) = 2$ and G^* has at least one vertex of degree zero, then G^* is one of the following graphs with a degree zero vertex added to it: one of the four trees of order five and maximum degree four, a cycle of order five together with a vertex of degree zero, a cycle of order four with a vertex split of degree one and a vertex of degree zero, or a cycle of order four together with a tree of order two. Since a cycle of order three is a triangle, we exclude those cases. The remaining graphs can be seen in Figure 7. Our goal is to show that in each case we can add back two of the four vertices we deleted back while maintaining planarity.
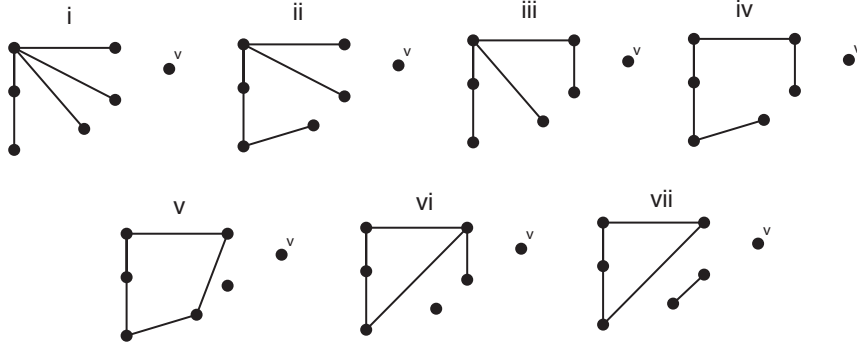

FIGURE 7. The seven triangle–free $(7, 5)$ graphs with at least one degree zero vertex and a maximum degree of four.

Since the vertices we delete from G to make G^* all have v as a neighbor, each one will be a vertex of degree three on the graph $G^* - v$. Hence adding one of these vertices, call it a, back keeps the planarity of G^*. Moreover, $G^* + a$ can be arranged such that at most one vertex of $G^* - v$ is not on the outer face and such a vertex, call it u, will have a degree of two in $G^* + a$. We have three more vertices from which to choose. If all were neighbors of u, then u would have a degree of five in G, which contradicts our degree sequence assumption. So we will be able to add two vertices back into our graph G^* while keeping its planarity. Hence if G has

the degree sequence $(4^9, 3^2)$ and does not contain a triangle, then it will be 2-apex, contradicting Lemma 1.2.

As we have encountered a contradiction for all possible degree sequences, this concludes the proof. □

7. 10 VERTEX GRAPHS

Proposition 7.1. *The only MMIK $(10, 21)$ graphs are the KS graphs, E_{10}, F_{10}, and H_{10}.*

Proof. Suppose G is a MMIK $(10, 21)$ graph. Since G is MMIK it has minimum degree at least three. As in Remark 2.1, we assume G is triangle–free and look for a contradiction.

Suppose we can delete two vertices from G, a and b, such that $\|G-a, b\| \leq 10$. If $G-a, b$ is non-planar it has either a $K_{3,3}$ minor of a K_5 minor. Since $\chi(G-a, b) \leq -2$ and $\delta(G-a, b) \geq 1$, then $G-a, b$ will have a tree of order two or three and by Lemma 2.2, there will be a triangle in G, a contradiction.

If there are $a, b \epsilon V(G)$, such that $\|G-a, b\| \leq 11$, then G is one of the eleven non-planar graphs in Figure 8. With the exception of *iii* in Figure 8, which has a tree of order two as a component leading to a triangle in G, each graph is a split $K_{3,3}$. Moreover, in each of these graphs, there are two adjacent original vertices whose neighborhoods are completely comprised of original vertices. Hence, by Lemma 2.4, adding a (or b) back into $G-a, b$ will either result in a 1–apex graph or will create a triangle. So, G is either 2-apex or has a triangle, a contradiction in either case.

Thus, it will be enough to consider cases where, for any $a, b \epsilon V(G)$, $\|G-a, b\| \geq 12$. With this constraint, the only possible degree sequences are $(5^6, 3^4)$, $(5^5, 4^2, 3^3)$, $(5^4, 4^4, 3^2)$, $(5^3, 4^6, 3)$, and $(5^2, 4^8)$. For the first four sequences, we realize that if all vertices of degree five are mutually adjacent, then we have a triangle. If not, then removing a and b of degree five and non-adjacent give $\|G-a, b\| = 11$, a case we considered above. This leaves only the $(5^2, 4^8)$ sequence.

Assume that G has the sequence $(5^2, 4^8)$ and recall that G contains no triangles. If the two degree five vertices (call them a and b) are not neighbors, then $\|G-a, b\| = 11$, so we will assume that a and b are neighbors. This means that a and b do not have any common neighbors, as otherwise, there would be a triangle. Hence, $G-a, b$ has the degree sequence (3^8) and, because G is triangle–free, $G-a, b$ is a bipartite graph with one part comprised of the neighbors of a in G and the other comprised of the neighbors of b in G. There is only one 3-regular bipartite graph with two parts of four vertices each. To see this, note that the "bipartite" complement (i.e. the edges of $K_{4,4}$ not present in $G-a, b$) is the disjoint union of four K_2's. Thus, $G-a, b$ is the cube and has a planar representation. Hence, G is 2-apex, a contradiction.

We conclude that the their are no triangle–free MMIK graphs with 21 edges and 10 vertices. Hence, the $(10, 21)$ MMIK graphs are the IK Heawood graphs of order ten, E_{10}, F_{10}, and H_{10}. □

8. GRAPHS THAT ARE NOT 2–APEX.

In this section we prove Propositions 1.4 and 1.5.

Proof. (of Proposition 1.4) Since a graph of 20 or fewer edges is 2–apex [Ma], the only N2A graph with $|G| \leq 7$ is K_7, which has no degree three vertices. So, the proposition is vacuously true for graphs of order seven or less.
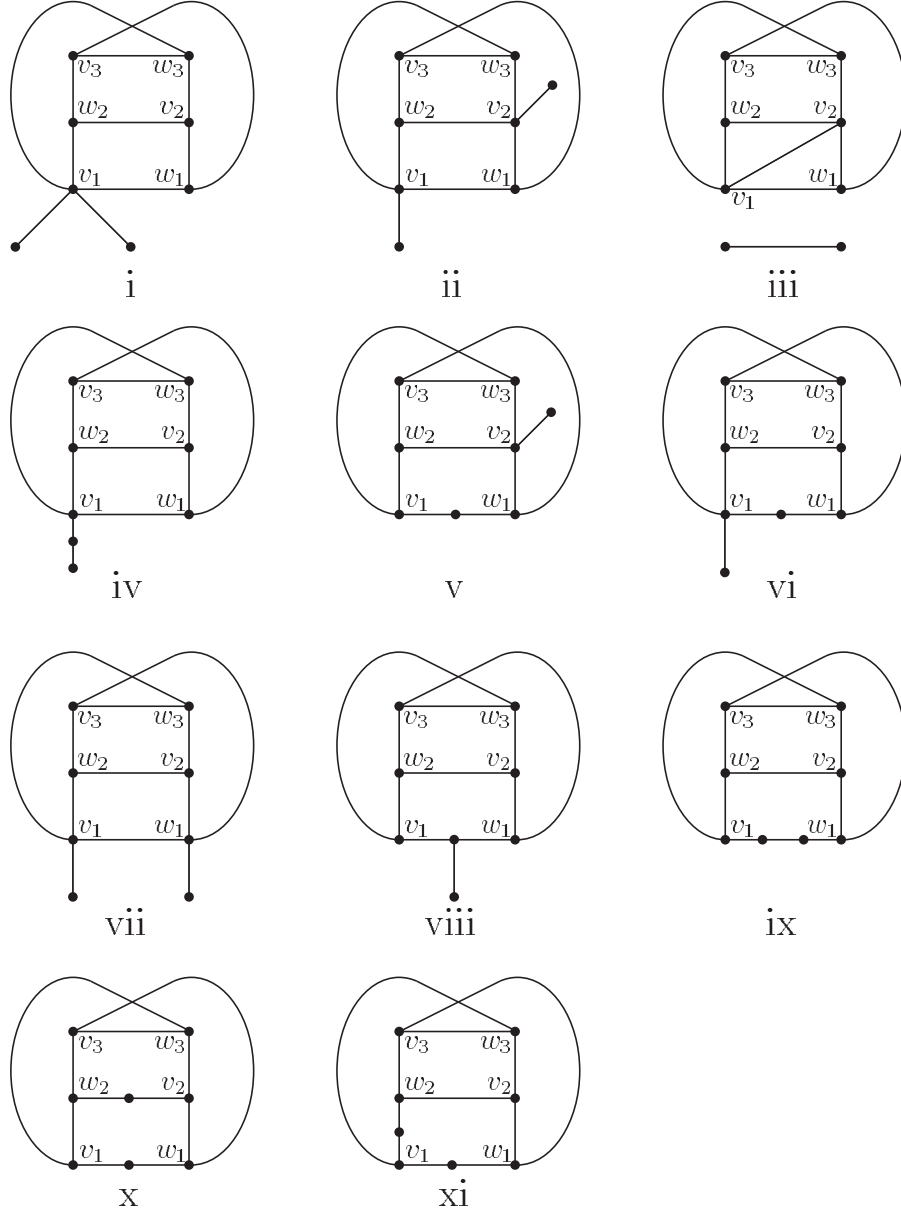
FIGURE 8. Non-planar graphs with eight vertices and eleven edges.

Suppose G is N2A with $|G| = 8$. As discussed in [Ma], G must be IK and we refer to the classification of such graphs due independently to [CMOPRW] and [BBFFHL]. There are 23 IK graphs on eight vertices, but only four have a vertex

of degree three. In each case, a $Y\nabla$ move on that vertex results in K_7, which is also N2A.

Again, graphs of size 20 or smaller are 2–apex. So, we can assume $\|G\| = 21$ and $|G| \geq 9$. If G is of order nine and N2A, then, by [Ma, Proposition 1.6], G is a Heawood graph (possibly with the addition of one or two isolated vertices). A $Y\nabla$ move results in the Heawood graph H_8 or $K_7 \sqcup K_1$, both of which are N2A.

This leaves the case where $|G| = 10$. Assume G is a $(10, 21)$ N2A graph that admits a $Y\nabla$ move to G'. For a contradiction, suppose G' is 2–apex with vertices a and b so that $G' - a, b$ is planar. Let v_0 be the degree three vertex in G at the center of the $Y\nabla$ move and v_1, v_2, v_3 the vertices of the resultant triangle in G'. Since G is N2A, it must be that $\{v_1, v_2, v_3\}$ is disjoint from $\{a, b\}$. Fix a planar representation of $G' - a, b$. The triangle $v_1 v_2 v_3$ divides the plane into two regions. Let H_1 be the induced subgraph on the vertices interior to the triangle and H_2 that of the vertices exterior. Then $|H_1| + |H_2| = 4$. Since G is N2A, there is an obstruction to converting the planar representation of $G' - a, b$ into a planar representation of $G - a, b$. This means that both H_1 and H_2 contain vertices adjacent to each of the triangle vertices $\{v_1, v_2, v_3\}$. In particular, H_1 and H_2 each have at least one vertex.

Suppose $|H_1| = |H_2| = 2$. The graph $G - b, v_1$ is non-planar, but, its subgraph $G - a, b, v_1$ is essentially a subgraph of $G' - a, b$ (with the addition of a degree two vertex v_0 on the edge $v_2 v_3$) and we will use the same planar representation for $G - a, b, v_1$ that we have for $G' - a, b$. Since $G - b, v_1$ is not planar, there's an obstruction to placing a in the same plane. If we imagine a outside of a disk that covers $G - a, b, v_1$, we see that their is some vertex in an H_i that is hidden from a. Without loss of generality, it's one of the vertices c_1 or d_1 of H_1, say c_1 that is inaccessible. This means we can assume that $c_1 v_2 d_1 v_3$ is a 4–cycle in G. However, as $G' - a, b$ is planar c_1 is also hidden from v_1 and $c_1 v_1$ is not an edge of the graph.

A similar argument using $G - b, v_2$ allows us to deduce a 4–cycle $c_2 v_1 d_2 v_3$ using the vertices c_2 and d_2 of H_2 while showing $c_2 v_2 \notin E(G)$. However, it follows that $G - b, v_3$ is planar, a contradiction.

So, we can assume $|H_1| = 3$ while H_2 consists of the vertex c_2 with $\{v_1, v_2, v_3\} \subset N(c_2)$. Suppose H_1 also has a vertex, c_1, that is adjacent to all three triangle vertices. As $G - b, v_1$ is non-planar, there's a vertex of H_1, call it d_1, that is hidden from a such that $c_1 v_2 d_1 v_3$ is a cycle in G and $d_1 v_1 \notin E(G)$. Similarly, $G - b, v_2$ shows that $c_1 v_1 e_1 v_3$ is in G and $e_1 v_2$ is not, e_1 being the third vertex of H_1. Now, $G - b, v_3$ will be planar unless $d_1 e_1 \in V(G)$. However, contracting $d_1 e_1$ shows that $G' - a, b$ has a $K_{3,3}$ minor and is non-planar, a contradiction.

If H_1 has no vertex c_1 that, on its own, is adjacent to the three triangle vertices, then either H_1 is connected, or else it is not but has an edge $c_1 d_1$ such that $\{v_1, v_2, v_3\} \subset N(c_1) \cup N(d_1)$. But, in this latter case, we can rearrange the planar representation of $G' - a, b$ such that the third vertex of H_1 is exterior to the triangle, returning to the earlier case where $|H_1| = |H_2| = 2$. So we will assume H_1 is connected.

Suppose H_1 is not complete, having only two edges $c_1 d_1$ and $d_1 e_1$. Again $G - b, v_1$ shows that at least two vertices of H_1 are in $N(v_2) \cap N(v_3)$ and there are two cases depending on whether or not $\{c_1, e_1\} \subset N(v_2) \cap N(v_3)$. If both c_1 and e_1 are in the intersection, then we can assume c_1 is hidden from a, meaning $a c_1 \in E(G)$, but $c_1 v_1 \notin E(G)$. Then $G - b, v_2$ shows that $d_1 v_1 e_1 v_3$ is in G and $e_1 v_2$ is not. But then

$G - b, v_3$ is planar, a contradiction. If c_1 and e_1 are not both in $N(v_2) \cap N(v_3)$, we can assume that c_1 and d_1 are the common vertices with at most one of those adjacent to v_1. If $c_1 v_1 \notin E(G)$, the argument is the same as above. So, we can assume it's d_1 that's hidden, meaning ad_1 is an edge and $d_1 v_1$ is not. In this case, $G - b, v_2$ must be planar, a contradiction.

Finally, if $H_1 = K_3$, then a similar sequence of arguments shows that, in G', the vertices of H_1 have neighborhoods as follows: $N(c_1) = \{a, b, d_1, e_1, v_2, v_3\}$, $N(d_1) = \{a, b, c_1, e_1, v_1, v_3\}$, and $N(e_1) = \{a, b, c_1, d_1, v_1, v_2\}$. By counting edges, we see that, in fact, a and b each have degree three and we have accounted for all edges in G'. Applying the ∇Y move to recover G, we observe that G is 2–apex (for example, $G - c_1, d_1$ is planar), a contradiction.

We've shown that assuming G' is 2–apex leads to a contradiction. Thus, the proposition also holds in the case $|G| = 10$, which complete the proof. $\square$

Proof. (of Proposition 1.5) Suppose G is MMN2A and $\|G\| = 21$. Note that $\delta(G) \geq 3$ as otherwise a vertex deletion or edge contraction on a small degree vertex gives a proper minor that is also N2A. This implies $|G| \leq 14$ and the case of $|G| = 14$ is Proposition 3.1. The cases where $|G| \leq 9$ are treated in [Ma]: a graph with $|G| \leq 8$ is N2A iff it is MMIK, so the proposition follows from the classification of MMIK graphs of order at most eight; and a graph with $|G| = 9$ is MMN2A if and only if it is one of the Heawood graphs E_9, F_9, or H_9.

This leaves the case where $|G| = 10$. If G has a degree three vertex, then apply a YT move at that vertex to get a graph G'. By Proposition 1.4 and the result of [Ma] for graphs of order nine, G' is Heawood, whence G is too. So, we can assume $\delta(G) \geq 4$ which means the degree sequence of G is $\{4^8, 5^2\}$ or $\{4^9, 6\}$.

Suppose there are vertices a and b such that $\|G - a, b\| = 11$. Then $G - a, b$ is one of the graphs of Figure 8. Since $\delta(G) = 4$, then $\delta(G - a, b) \geq 2$. so $G - a, b$ is one of graphs ix, x, and xi in the figure. In all three cases, both a and b must be adjacent to both v_3 and w_3. For if, for example, a and v_3 are not adjacent, then $G - b, w_3$ is planar. This means v_3 and w_3 have degree five in G, which contradicts the two given degree sequences for G. We conclude there is no choice a and b such that $\|G - a, b\| = 11$.

This means G must have degree sequence $\{4^8, 5^2\}$ with the two vertices of degree five adjacent and $G - a, b$ a $(8, 12)$ graph. There are two cases depending on whether or not a and b have a common neighbor in G. Suppose first that c is adjacent to both a and b. In $G - a, b$ vertex c will have degree two and we can use D2 to delete c, arriving either at a $(7, 11)$ graph or else a multigraph with a doubled edge. Removing the extra edge if needed, let H denote the resulting $(7, 11)$ or $(7, 10)$ graph.

If H is $(7, 10)$, it is one of the two graphs of Figure 9. In the case of the graph on the left, the doubled edge must be that incident on the degree one vertex as $\delta(G - a, b) \geq 2$. But then the vertex labelled v_1 in the figure will have degree five in $G - a, b$, contradicting our assumption that a and b were the only vertices of degree greater than four. So, we can assume H is the graph to the right in the figure. Up to symmetry, the doubled edge of H is either uv_1, $v_1 w_2$, or $v_2 w_2$. We'll examine the first case; the others are similar. Doubling uv_1 and adding back c leaves v_1 of degree four in $G - a, b$. Then $G - a, b, v_1$ simplifies to $K_{3,3} - v_1$. Since w_1, w_2, and w_3 all have degree three in $G - a, b$, they each have exactly one of a and b as a neighbor in G. Suppose a is adjacent to w_2. Then $G - a, v_1$ is planar, contradicting
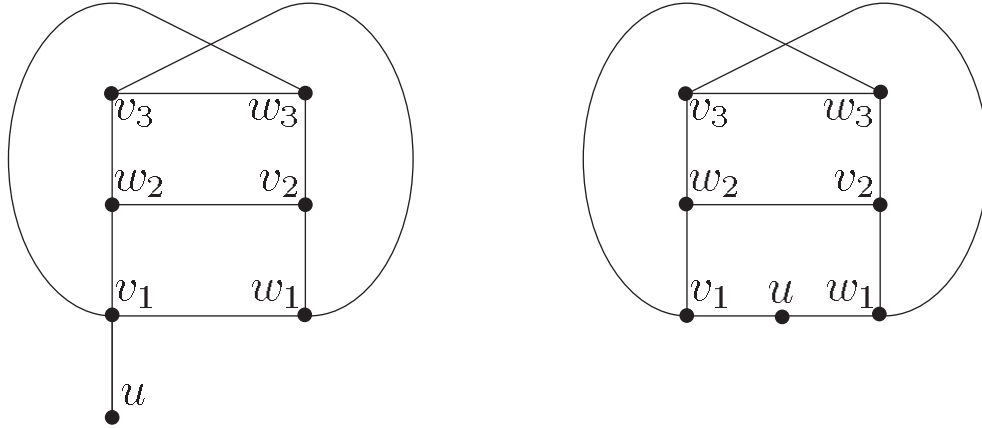
FIGURE 9. The two non-planar (7,10) graphs of minimal degree at least one.

G being N2A. For the other two choices of edge doubling, once can again delete a resulting degree four vertex along with a or b to achieve a planar graph. So H being $(7, 10)$ leads to a contradiction.
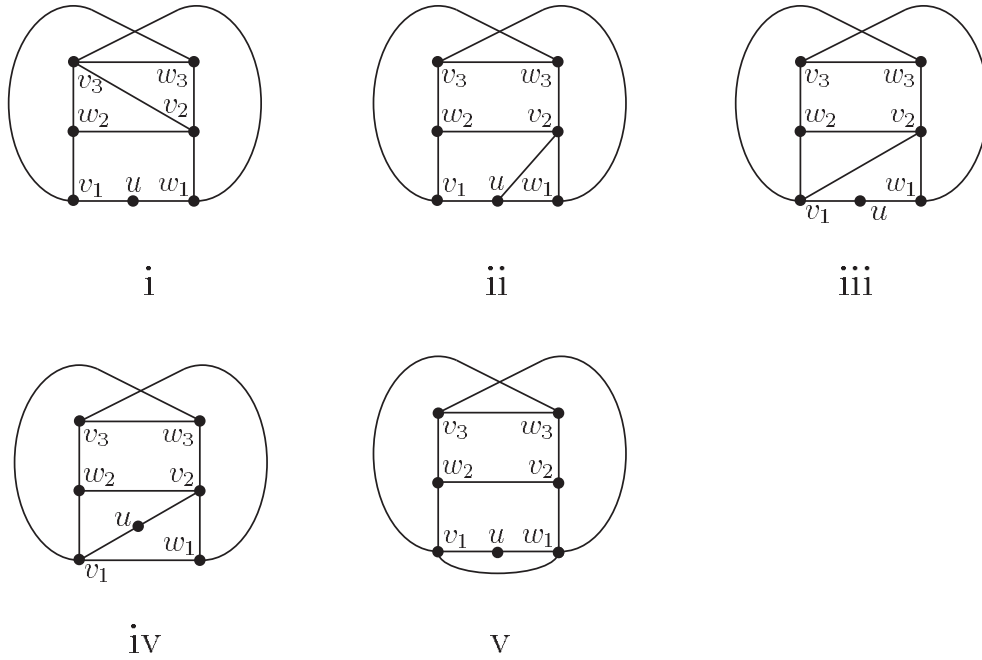

FIGURE 10. The five non-planar (7,11) graphs of minimal degree at least two.

If H is $(7, 11)$, then $\delta(H) = \delta(G - a, b) \geq 2$ and H is one of the five graphs of Figure 10. Here we use a similar approach. Deleting one of the degree four vertices

of H, call it x, results in a graph $G - a, b, x$ that simplifies to $K_{3,3} - v_1$. Since each of the degree three vertices of H is adjacent to exactly one of a and b, there will be an appropriate choice from those two, say a, such that $G - a, x$ is planar, which is a contradiction. So, H being $(7, 11)$ is not possible and we conclude that there is no such vertex c that is adjacent to both a and b.
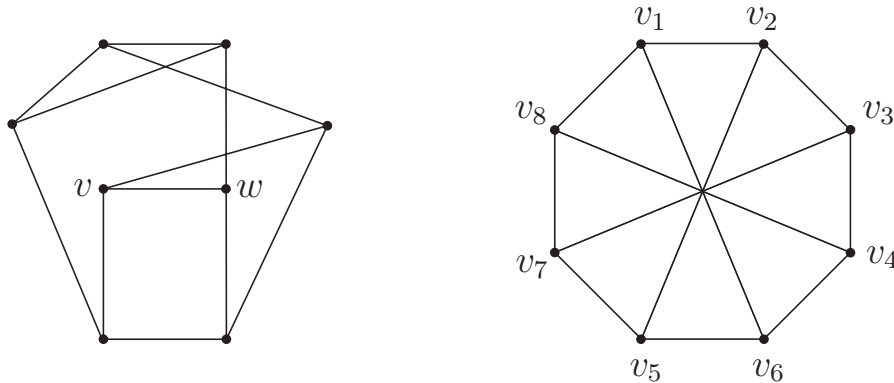

FIGURE 11. The two non-planar cubic graphs of order eight

This means that $G - a, b$ is a non-planar cubic graph (i.e., 3-regular) on eight vertices. There are two such graphs, shown in Figure 11. If $G - a, b$ is the graph to the left in Figure 11, note that the vertex labelled v is adjacent to exactly one of a and b, say a. Then $G - a, w$ is planar.

Finally, assume that $G - a, b$ is the graph to the right in Figure 11. Note that each vertex of $G - a, b$ is adjacent to exactly one of a and b in G. If a and b are adjacent to alternate vertices in the 8–cycle (for example if $\{v_1, v_3, v_5, v_7\} \subset N(a)$ and $\{v_2, v_4, v_6, v_8\} \subset N(b)$), we obtain graph 20 of figure 1, a Heawood graph. If not, then we must have two consecutive vertices, say v_1 and v_2 that share the same neighbor in $\{a, b\}$, say a. That is, we can assume $av_1, av_2 \in E(G)$. Then $G - a, v_3$ is planar, contradicting G being N2A.

In summary, if G of order 10 is N2A with $\delta(G) > 3$, it must be graph 20 of the Heawood family. This completes the proof of Proposition 1.5. $\square$

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